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The vibrating string controversy

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In the mid-1700s a debate raged between Jean d'Alembert, Leonhard Euler, and Daniel Bernoulli concerning the proper solution to the classical wave equation. This controversy was partially solved by Lagrange and, more conclusively, by Fourier (50 years later) and it provides an interesting case study for the role of mathematics in the modeling of physical phenomena. Of particular note in this debate, was the meaning of boundary conditions. The controversy is summarized from the point of view of this mathematical physics perspective.

INTRODUCTION

Mathematical descriptions of wave phenomena are fundamental to many areas of physics. A clear understanding of the relations which describe the vibrating string is required to comprehend more complex wave motions. Few physicists, however, are aware of the intense controversy that existed over the original descriptions of the vibrating string proposed during the eighteenth century. At the height of the controversy one of the most fundamental and powerful theorems of mathematical physics emerged, was overlooked, and had to wait 50 years for its rediscovery.

While the debate has long held interest for mathematical historians, there has been little discussion of the way this debate signaled the emergence of a new kind of physicist. There are excellent reviews of the controversy,[1,2] each presented as a topic from the history of mathematics. In presenting our view of the debate, we have drawn extensively from these sources, as well as the original papers.

Physicists will find the controversy enlightening. Many of the principles of applying mathematical formalism to physical phenomena which we take for granted today were poorly understood at the time of the debate. In particular, the extent to which a mathematical formalism should explain a physical phenomena had yet to be fully established during this post-Newtonian period of rational mechanics. The resulting confusion contributed to the antagonism between the participants.

This unclear connection between a physical phenomena and its mathematical description suggests another perspective one may take of the controversy. In our view, the debate is a dispute between a mathematician, a traditional physicist, and a newly emerging scholar: the mathematical physicist.

THE PRINCIPAL PARTICIPANTS

Jean le Rond d'Alembert made significant contributions to physics which include treatises on motion in resistive media, music, the three-body problem, and the precession of the equinoxes.[3] He is probably best known by physicists for his observation that the sum of the internal forces of a rigid body must vanish (d'Alembert's principle). His mathematical accomplishments, however, distinguish him as the most eminent French mathematician of the mid-eighteenth century.[4] He is, therefore, "the mathematician" in our presentation of the controversy. d'Alembert was the first member of the debate to publish a study on the motion of a vibrating string.[5] He derived the partial differential equation[6]

$$\frac{\partial^2 y(x,t)}{\partial t^2} = \frac{\partial^2 y(x,t)}{\partial x^2} \tag{1}$$

and constructed a general solution consisting of two arbitrary functions f and g:

$$y(x,t) = f(x+t) + g(x-t) .$$

Applying the boundary conditions $y(0,\ t) = y(L,\ t)$ d'Alembert reduced his solution to

$$y(x,t) = f(x+t) + f(x-t) .$$

The only restrictions d'Alembert felt necessary to impose on the arbitrary function f were that it be periodic, odd,[7] and *everywhere differentiable*. This latter restriction was required since $f(x \pm t)$ is just $f(x)$ translated to the left or right by an amount t. d'Alembert was concerned only with a general mathematical solution to the partial differential equation (1) and had no interest in any physical significance of the interval $0 \leqslant x \leqslant L$. Indeed, it was not unusual for d'Alembert to sacrifice physical reality for mathematical purity.[8]

There can be no doubt that Leonhard Euler was one of the most productive mathematicians of all times. During his life he published more than 500 articles and books, averaging over 800 pages a year. In addition to contributions in every field of mathematics known during his time, he was responsible for generating a large portion of present day mathematical notation. Euler's first independent contribution to science came at the age of 19 when he won an honorable mention from the Paris Academy of Science for a paper on shipmasting. Prior to this, he had shared an award with Daniel Bernoulli and Colin Maclaurin for a paper they jointly authored on the tides. Euler was the first to publish a mechanics textbook which applied the full analytic power of the calculus to Newtonian dynamics. Later, he produced another mechanics text which introduced the Eulerian equations for rigid body rotations.

All of these analyses of physical problems, his demonstrated mathematical prowess, and—most important—his stand during the vibrating string debate, permit us to view Euler as the "mathematical physicist" in the controversy. It must be added, however, that when Euler's life is viewed outside the context of this controversy, it is not so easy to apply this label. Very often, in fact, physics appears to have been nothing more for Euler than a starting point for a rapid move to more pure (and less applied) mathematics.

Euler's original analysis of the vibrating string problem appeared in two papers of identical content (the first in French and the second in Latin) in 1748 and 1749.[9] He derived the more general wave equation

$$\frac{1}{c^2}\frac{\partial^2 y(x,t)}{\partial t^2} = \frac{\partial^2 y(x,t)}{\partial x^2} , \tag{2}$$

found the solution

$$y(x,t) = f(x+ct) + g(x-ct) ,$$

and applied the boundary conditions to yield

$$y(x,t) = f(x+ct) + f(x-ct) . \tag{3}$$

Euler differed from d'Alembert on the specification of the function f. He claimed that f could be deduced solely from initial conditions: If $Y(x)$ and $V(x)$ are the initial position and velocity of the string, then

$$y(x,t) = \frac{1}{2}\left[Y(x+ct) + Y(x-ct) + \frac{1}{c}\int_{x-ct}^{x+ct} V(s)ds \right] . \tag{4}$$

Further, Euler proclaimed that the functions $Y(x)$ and $V(x)$ need not be functions in the ordinary sense, but may by any curve *drawn by hand* in the interval $0 \leqslant x \leqslant L$ and extended along the real line with odd periodicity. Euler, "the mathematical physicist," clearly had in mind the plucked string; he was using a physical observation to impose a mathematical condition. When he permitted these new curves with corners to be solutions to the wave equation, the controversy was on its way.

Daniel Bernoulli was unmistakably the traditional physicist in the controversy. Coming from a distinguished family of mathematicians, he is credited with numerous contributions to fluid dynamics. However, his mathematical abilities did not compare with those of d'Alembert or Euler. A note of caution is appropriate here: It would be a mistake within the context of the controversy to view Bernoulli's contributions as anything less than remarkable. At the time of this debate Bernoulli had a position of medicine at Bale. Bernoulli's approach to the vibrating string problem smacks of a physicist; his solution is based entirely upon the physical phenomena of the vibrating string. He asked his readers to listen to the string! His analysis was probably based on Brook Taylor's previous observation of the wave's fundamental amplitude:

$$y(x) = A \sin(\pi x/L) .$$

Bernoulli argued that the solution must be a sum of this fundamental and higher harmonics:

$$y(x,t) = A_1 \sin(\pi x/L)\cos(\pi ct/L) + A_2 \sin(2\pi x/L)\cos(2\pi ct/L) + \cdots . \tag{5}$$

The principle of superposition was unknown at this time, so Bernoulli devoted a great amount of space in his two papers of 1753 to attempts at justifying this sum.[10] He examined the oscillatory motions of a system of several particles, but provided no mathematical support for his arguments. In the end, he could suggest no method by which the coefficients A_1, A_2..., in Eq. (5) might be evaluated.

THE CONTROVERSY

Since there are, at this point, three participants in the debate, the actual dates of the ensuing replies and retorts to replies are somewhat entangled. (Also, very often there was a considerable time, occasionally years, between composition and publication.) For clarity, we will not closely adhere to the chronological order of the various publications, but will focus attention on the important content of the arguments.

D'ALEMBERT VERSUS EULER

While d'Alembert objected to Euler's new functions with corners, his initial response during the late 1750s (actually published in 1773), was more restrained than that of a later paper. In this initial criticism of Euler's approach, he restated his opinion that the function f [in Eq. (3)] must be periodic, odd, and everywhere differentiable.[11] This paper is important to mathematical physics since d'Alembert rederived his solution by a new technique, which was the first application of the method of separation of variables.

His 1761 response was a genuine attack on Euler.[12] He objected to Euler's use of physical arguments for the admission of his new functions. He pointed out that the very approximation which permits the derivation of the wave equation (small displacements from equilibrium) *prohibits the use of physical arguments.* He continued his assault on mathematical grounds by questioning the interpretation of the wave equation for a plucked string (Fig. 1).

d'Alembert noted that in the neighborhood of the point p (at which the string is plucked) the right and left slopes are unequal:

$$\left.\frac{\partial y}{\partial x}\right|_{p+} \neq \left.\frac{\partial y}{\partial x}\right|_{p-} .$$

Hence, $(\partial^2 y)/(\partial x^2)$ is undefined. And, he asks, if it is undefined, how can it equal $(1/c^2)(\partial^2 y)/(\partial t^2)$?

d'Alembert continued: Since $y(x,t)$ is determined (as Euler claimed) from this function extended over the whole real line [Eq. (4)], there are times when the wave equation is undefined at any point on the string!

Euler's replies to d'Alembert's criticisms were weak and presented in papers of 1762 and 1765.[13,14] In essence, his arguments reduce to the assertion that since the displacements are small, the curve at a corner point deviates only infinitesimally from a differentiable function. He maintained that any error introduced was so smalled as to be "entirely nothing." d'Alembert, of course, was not convinced.

D'ALEMBERT AND EULER VERSUS BERNOULLI

d'Alembert objected to Bernoulli's series solution on physical grounds.[11] For those of us who have principle of superposition well engraved in our view of the vibrating string, his objections can seem somewhat mystifying. d'Alembert did not view any motion of the string as a compound motion of separate distinct modes. Instead, he felt that there was one and only one frequency associated with a vibrating condition. Hence, a trigonometric decomposition was not appropriate since it represented a multiple of frequencies.

Euler's criticisms of Bernoulli's solution were directed toward the algebraic properties of his solution. Euler felt that this trigonometric series could not possibly be general enough to represent an arbitrary function f, whether or not that function be one drawn by hand or a function in the traditional (d'Alembert) sense. Euler, our mathematical physicist, especially disputed the notion that Bernoulli's series could represent the wave form generated from the initial condition of "snapping" the string at one end (see Fig. 2).[14]

It is within Euler's arguments that a major mathematical misconception originated, one which none of the participants were ever able to surmount. No one realized that the solution need only be for the interval $0 \leqslant x \leqslant L$. What happens outside is irrelevant. Although Euler and d'Alembert disagreed over the degree of generality for the functions which could constitute a solution, they both agreed that the inherent properties of the trigonometric functions were sufficiently restrictive so as to exclude their use to represent any arbitrary function. The problem was periodicity. Eighteenth century mathematicians understood the intrinsic periodicity of the trigonometric functions. However, no one perceived that the periodicity established by specifying a function $f(x)$ in the interval $0 \leqslant x \leqslant L$ was a *geometrical* periodicity defined within the interval. Euler argued that Bernoulli's series was odd (in x) and all functions are not odd. We recognize this argument as irrelevant since the function can be made odd or even, depending upon how it is extended along the negative x axis.

This confusion over periodicity caused Euler to miss an effective attack he could have made on Bernoulli's solution.[15] If his solution [Eq. (5)] is differentiated to yield the transverse velocity function, at $t = 0$ this series is zero. Hence, Bernoulli's solution is only appropriate for zero initial string velocity. This is less general than Euler's own solution.

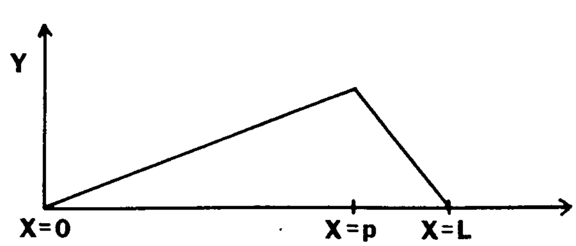

Fig. 1. Euler's "plucked" string.

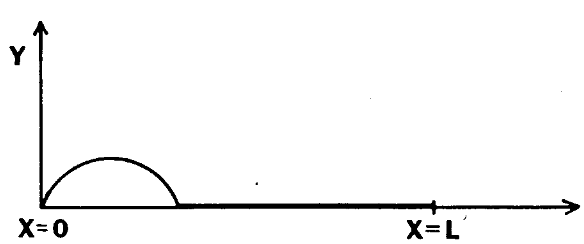

Fig. 2. Euler's "snapped" string.

At this point we have d'Alembert objecting to physical arguments for solutions to a partial differential equation, and calling for the other participants to engage in mathematics. Euler is defending his new functions with corners but relying on vague infinitesimal arguments to support his case. Finally, Bernoulli is asking the others to listen to the string, but providing no mathematical support for his arguments. And then, a fourth participant joins the controversy with a completely new approach.

LAGRANGE: ANOTHER MATHEMATICAL PHYSICIST ENTERS THE DEBATE

Luigi de la Grange Tournier entered the debate in 1759 with a lengthy paper on sound propagation.[16] For his approach to the problem, and the many other lasting contributions he made to physics, we are inclined to distinguish him also as a mathematical physicist.

Lagrange's analysis of the vibrating string is contained within his paper on sound propagation. He reviewed the derivation of the wave equation, and the arguments of d'Alembert, Euler, and Bernoulli. He supported Euler's solution determined from the initial conditions, and the admission of the new functions with corners. However, Lagrange objected to Euler's unclear use of infinitesimals. He dismissed Bernoulli's solution, endorsing Euler's arguments against the generality of the trigonometric series. He, too, fell into the periodicity trap.

Lagrange's approach to the vibrating string problem completely avoided the wave equation. He constructed the string from a collection of n, equally spaced, point masses, connected by a light cord. This yielded a set of n equations of the form[17]

$$\frac{d^2y_k}{dt^2} = c^2(y_{k-1} - 2y_k + y_{k+1})$$

Lagrange solved the system of equations first for a finite number of masses. Next he generated the solution for the vibrating string by allowing the number of masses to become infinite as the spacing between each decreased to zero. He found[18]

$$\begin{aligned} y(x,t) = \frac{2}{L}\int_0^L dX\,Y(X)\Big[&\sin\Big(\frac{\pi X}{L}\Big)\sin\Big(\frac{\pi x}{L}\Big)\cos\Big(\frac{\pi ct}{L}\Big) \\ &+ \sin\Big(\frac{2\pi X}{L}\Big)\sin\Big(\frac{2\pi x}{L}\Big)\cos\Big(\frac{2\pi ct}{L}\Big) + \cdots\Big] \\ + \frac{2}{\pi c}\int_0^L dX\,V(X)\Big[&\sin\Big(\frac{\pi X}{L}\Big)\sin\Big(\frac{\pi x}{L}\Big)\cos\Big(\frac{\pi ct}{L}\Big) \\ &+ \frac{1}{2}\sin\Big(\frac{2\pi X}{L}\Big)\sin\Big(\frac{2\pi x}{L}\Big)\cos\Big(\frac{2\pi ct}{L}\Big) + \cdots\Big], \end{aligned}$$

where $Y(x)$ and $V(x)$ are the initial position and velocity of the string. This result brought Lagrange very close to the Fourier series. (Fourier was born in 1768.) If the initial displacement $Y(x)$ is substituted into Lagrange's solution for $t = 0$, and the order of the integral and sum are reversed, we have

$$Y(x) = \frac{2}{L}\sum_{n=1}^{\infty}\Big[\int_0^L Y(X)\sin\Big(\frac{n\pi X}{L}\Big)dX\Big]\sin\Big(\frac{n\pi x}{L}\Big).$$

This is an odd Fourier series.

Grattan-Guinness suggests that there are at least three reasons Lagrange did not discover the series.[19] First, he was not looking for a theorem of mathematical analysis. He was concerned with the vibrating string as a problem related to the nature of sound propagation. Second, Lagrange was seeking an integral solution, and not an infinite series result. This also is a consequence of his view of the string's motion in the context of a theory of sound. Finally, the series representation would have been Bernoulli's solution, with coefficients given by the integrals. And, of course, Lagrange did not feel the trigonometric series was sufficiently general. Ravetz notes that during this period several mathematicians came very close to the Fourier series.[20] However, all apparently viewed it as a very accurate interpolation mechanism.

Euler felt Lagrange's solution supported his own arguments for the generality of the initial conditions. Lagrange agreed but considered his methods preferable since they avoided Euler's infinitesimal arguments. (However, Lagrange clearly could not escape the use of infinitesimals.)

With the appearance of Lagrange's paper, the intensity of the debate significantly declined. Too much mathematics had yet to be clarified for meaningful progress to be made. More partial differential equations were derived, geometrical interpretations of their solutions began to evolve, and the theory of functions was refined. Finally, in 1807, Joseph Fourier was led to his famous series while studying the heat diffusion equation. Lagrange was the only living member of the debate. He objected to Fourier's results, questioning the convergence of the series. Fourier was eventually able to demonstrate the convergence of at least one of his solutions. The subsequent mathematical rigor required to determine the conditions under which the general series would converge had a major influence on nineteenth century mathematics.

CONCLUSIONS

The vibrating string controversy was an important episode in the history of physics. In addition to founding the analysis for many theories of wave motion, it helped focus attention toward the relation of mathematics to physical theory. It is interesting that in retrospect, the solutions proposed by each of the participants were correct: d'Alembert's differentiable functions of $x \pm ct$ are general solutions to the wave equation. Euler's solutions determined by the initial conditions are perfectly valid. Fourier has shown that Euler's new functions can be represented by a trigonometric series. And, finally, Bernoulli's solution is a special case of the trigonometric series for zero initial velocity.

[1] J. R. Ravetz, "Vibrating Strings and Arbitrary Functions," in *The Logic of Personal Knowledge* (The Free Press, Glencoe, IL, 1961), pp. 71–88.

[2] I. Grattan-Guinness, *The Development of the Foundations of Mathematical Analysis from Euler to Riemann* (MIT, Cambridge, MA, 1970), Chap. 1.

[3] Thomas L. Hankins, *Jean d'Alembert Science and the Enlightenment* (Clarendon, Oxford, 1970).

[4] Carl B. Boyer, *A History of Mathematics* (Wiley, New York, 1968), pp. 489–498.

[5] J. d'Alembert, "Recherches sur la courbe que forme une corde tendue mise en vibration," Mem. Acad. Sci. Berlin **3**, 214–219 (1747).

[6] The differential equation and its solution might be a little unfamiliar to readers of modern texts. d'Alembert considered only a unitary wave speed so the *c*s are missing in his paper.

[7] An equivalent, alternative form of d'Alembert's solution shows more clearly that this function is odd: $y(x,t) = f(t + x) - f(t - x)$.

[8] Hankins, Ref. 3, p. 46.

[9] Leonhard Euler, "Sur la vibration des cordes," Mem. Acad. Sci. Berlin

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[10]Daniel Bernoulli, "Reflexions et eclairissemens sur les nouvelles vibrations des cordes exposees dans les, Memorires de l'Academie de 1747 et 1748, Mem. Acad. Sci. Berlin **9**, 147–172 (1755); "Sur le melange de plusieurs especes de vibrations simples isochrones, qui peuvent coexixter dans une meme systeme de corps," *ibid.*, pp. 173–195.

[11]J. d'Alembert, "Fondamental," in *Encyclopedie Ou Dictionnaire Raisonne Des Sciences, Des Arts Et Des Metiers* 7, 52–53 (1773).

[12]J. d'Alembert, "Recherches sur les vibrations des cordes sonores," in *Pamphlets Opuscules Mathematiques* **1**, 1–73 (1761).

[13]L. Euler, "Eclaircissemens sur le mouvement des cordes vibrantes," in *Works*, Series 2, 10, pp. 337–396.

[14]L. Euler, "Eur le mouvement d'une corde, qui au commencement n'a ete ebranlee que dans une partie," in *Works*, Series 2, 10, pp. 426–450.

[15]I. Grattan-Guinness, Ref. 2, p. 12.

[16]J. Lagrange, "Recherches sur la nature et la propagation du son," Miscell. Taurin. **1**, 39–148 (1759).

[17]J. Marion, *Classical Dynamics of Particles and Systems*, 2nd ed. (Academic, New York, 1970), pp. 438.

[18]The "dx" in this equation has been substituted for Lagrange's "dx." The latter makes no sense mathematically nor in the context of his paper. However, it is not a typographical error. The point is discussed by Grattan-Guinness, Ref. 2, pp. 16 and 17.

[19]I. Grattan-Guinness, Ref. 2, pp. 16 and 17.

[20]J. Ravetz, Ref. 1, p. 85.

The fly ball trajectory: An older approach revisited

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An older approach to the problem of projectile motion with quadratic drag force is presented with the fly ball as an example. In this approach, analytical solutions for the velocity, curvature, and arc length are obtained as functions of the slope angle. It is shown that the velocity and curvature do not have their extrema at the top of the trajectory but during the early phase of descent. The entire problem is reduced to simple integrations over the slope angle.

I. INTRODUCTION

It has been widely stated that the motion of a projectile under a drag force proportional to the square of the velocity has no analytical solution and therefore must be tackled numerically.[1–4] With the advent of desk computers, various numerical schemes have been developed to solve coupled equations of motion in x and y and the solutions have been applied to find the range of the shot put,[2] golf ball,[3] and baseball.[4] This has somewhat obscured the fact that, in the absence of wind, intrinsic solutions of the problem exist in terms of the slope angle θ of the velocity vector v.[5–10] Expressions for v can be found in several standard textbooks,[5–8] and so are discussions on more general forms of the drag force.[8–10] Moreover, the entire problem of finding the horizontal distance x, vertical distance y, and time t can be reduced to much simpler integrations over θ.[6,9,11] In this paper, we revisit this older approach with reference to the trajectory of a fly ball leaving the bat at a speed of 100 mph at an angle of 60° to the horizontal.[4]

II. EXACT SOLUTIONS FOR VELOCITY, CURVATURE, AND ARC LENGTH

Since air resistance is antiparallel to the velocity, the equations of motion can be written as

$$m\frac{dv}{dt} = -mg\sin\theta - cv^2 \tag{1}$$

and

$$\frac{mv^2}{\rho} = mg\cos\theta, \tag{2}$$

where m is the mass of the projectile, g the acceleration due to gravity, and c the quadratic drag coefficient. The radius of curvature ρ of the trajectory is given by

$$\rho = -\frac{ds}{d\theta} = -\frac{ds}{dt}\frac{dt}{d\theta} = -v\frac{dt}{d\theta}. \tag{3}$$

Substitution of Eq. (3) in Eq. (2) gives

$$v\frac{d\theta}{dt} = -g\cos\theta. \tag{4}$$

Eliminating t between Eqs. (1) and (4), we get,

$$\frac{dv}{d\theta} - v\tan\theta = \frac{c}{mg}v^3\sec\theta. \tag{5}$$

This is a particular case of Bernoulli's equation.[12] Following Timoshenko and Young,[7] Eq. (5) can be conveniently rewritten as

$$\frac{d(v\cos\theta)}{(v\cos\theta)^3} = \frac{c}{mg}\frac{d\theta}{\cos^3\theta}. \tag{6}$$

Integrating both sides, we get,

$$\frac{1}{(v\cos\theta)^2} = -\frac{c}{mg}[\ln(\sec\theta + \tan\theta) + \sec\theta\tan\theta] + C. \tag{7}$$

